

TRANSFER OPERATORS, INDUCED PROBABILITY SPACES, AND RANDOM WALK MODELS

PALLE JORGENSEN AND FENG TIAN

ABSTRACT. We study a family of discrete-time random-walk models. The starting point is a fixed generalized transfer operator R subject to a set of axioms, and a given endomorphism in a compact Hausdorff space X . Our setup includes a host of models from applied dynamical systems, and it leads to general path-space probability realizations of the initial transfer operator. The analytic data in our construction is a pair (h, λ) , where h is an R -harmonic function on X , and λ is a given positive measure on X subject to a certain invariance condition defined from R . With this we show that there are then discrete-time random-walk realizations in explicit path-space models; each associated to a probability measures \mathbb{P} on path-space, in such a way that the initial data allows for spectral characterization: The initial endomorphism in X lifts to an automorphism in path-space with the probability measure \mathbb{P} quasi-invariant with respect to a shift automorphism. The latter takes the form of explicit multi-resolutions in L^2 of \mathbb{P} in the sense of Lax-Phillips scattering theory.

CONTENTS

1. Introduction	1
2. The Setting	2
3. Iterated Function Systems: The General Case	9
4. The Set $\mathcal{L}_1(R)$ from a Quadratic Estimate	10
5. From Endomorphism to Automorphism	12
5.1. Multi-Resolutions	15
6. Harmonic Functions from Functional Measures	16
References	17

1. INTRODUCTION

We study a family of stochastic processes indexed by a discrete time index. Our results encompass the more traditional random walk models, but our study here goes beyond that. The processes considered are generated by a single positive operator, say R , defined on $C(X)$ where X is a given compact Hausdorff space. From a given positive operator R we then derive an associated system of generalized transition probabilities, and an induced probability space; the induction realized as a probability space of infinite paths having X as a base space. In order

2000 *Mathematics Subject Classification.* Primary 47L60, 46N30, 65R10, 58J65, 81S25.

Key words and phrases. Unbounded operator, closable operator, spectral theory, discrete analysis, distribution of point-masses, probability space, stochastic processes, discrete time, path-space measure, endomorphism, harmonic functions.

for us to pin down the probability space, i.e., the induced path-space measure \mathbb{P} , two more ingredients will be needed, one is a prescribed endomorphism σ in X , consistent with R ; and, the other, is a generalized harmonic function h on X , i.e., $R(h) = h$. We further explore the interplay between the harmonic functions h and the associated path-space measures \mathbb{P} . To do this we note that the given endomorphism σ in X induces an automorphism in the path-space. We show that the path-space measure \mathbb{P} is quasi-invariant, and we compute the corresponding Radon-Nikodym derivative. Our motivation derives from the need to realize multiresolution models in a general setting of dynamical systems as they arise in a host of applications: in symbolic dynamics, e.g., [BJO04, BJKR02], in generalized multiresolution model, e.g., [DJ09]; in dynamics arising from an iteration of substitutions, e.g., [Bea91]; in geometric measure theory, and for Iterated Function Systems (IFS), e.g., [Hut81, Urb09]; or in stochastic analysis, e.g., [AJ12, Hut81, MNB16, GF16, ZXL16, YPL16, TSI⁺15, Pes13, KLTMV12].

2. THE SETTING

Organization of the paper: The setup starts with a fixed and given compact Hausdorff space X , and positive operator R (a generalized transfer operator), defined on the function algebra $C(X)$, and subject to two simple axioms. Candidates for X will include compact Bratteli diagrams, see e.g., [Mat06, Mat04, Dan01, HPS92]. Given R , in principle one is then able to derive a system of generalized transition probabilities for discrete time processes starting from points in X ; see details in the present section. However, in order to build path-space probability spaces this way, more considerations are required, and this will be explored in detail in Sections 4 and 5 below. Section 3 deals with a subfamily of systems where the generalized transfer operator R is associated to an Iterated Function system (IFS). In Section 6, we derive some conclusions from the main theorems in the paper.

Let X be a compact Hausdorff space, and $\mathcal{M}(X)$ be the space of all measurable functions on X . Let $R : C(X) \rightarrow \mathcal{M}(X)$ be a positive linear mapping, i.e., $f \geq 0 \implies Rf \geq 0$.

Definition 2.1. Let $\mathcal{L}(R)$ be the set of all positive Borel measures λ on X s.t.

$$R(C(X)) \subset L^1(\lambda), \text{ and} \tag{2.1}$$

$$\lambda \cdot R \ll \lambda \text{ (absolutely continuous)}. \tag{2.2}$$

Note 2.2. Let R be as above, and set

$$\mu = \lambda \cdot R, \text{ and } W = \frac{d\mu}{d\lambda} = \text{Radon-Nikodym derivative, then} \tag{2.3}$$

$$\lambda(Rf) \stackrel{\text{defn.}}{=} \underbrace{\int_X Rf d\lambda}_{\mu(f)} = \int_X f W d\lambda, \quad \forall f \in C(X). \tag{2.4}$$

W depends on both R and λ .

Definition 2.3. For all $x \in X$, let

$$\mu_x = P(\cdot | x)$$

be the conditional measure, where

$$\mu_x(f) := (Rf)(x) = \int_X f(y) d\mu_x(y). \tag{2.5}$$

Lemma 2.4. *Let X , and R (positive in $C(X)$) be as before, then there is a system of measures $P(\cdot | x)$ such that*

$$(Rf)(x) = \int_X f(y) P(dy | x), \quad \forall f \in C(X). \quad (2.6)$$

Proof. Immediate from Riesz' theorem applied to the positive linear functional,

$$C(X) \ni f \longrightarrow R(f)(x), \quad \forall x \in X.$$

□

Corollary 2.5. *$C(X) \ni f \longrightarrow R(f)(x)$ extends to $F \in \mathcal{M}(X)$, measurable functions on X , s.t. the extended operator \tilde{R} is as follows:*

$$\tilde{R}(F)(x) = \int_X F(y) P(dy | x), \quad F \in \mathcal{M}(X). \quad (2.7)$$

We will write R also for the extension \tilde{R} .

Remark 2.6. Let X , and R be as specified in Definition 2.1. Set

$$\mathcal{L}_1(R) = \{\lambda \in \mathcal{L}(R) \mid \lambda(X) = 1\}.$$

Clearly, $\mathcal{L}_1(R)$ is convex. In this generality, we address two questions:

- Q1. We show that $\mathcal{L}_1(R)$ is non-empty.
- Q2. What are the extreme points in $\mathcal{L}_1(R)$?

Lemma 2.7. *Let $\mu_x = P(\cdot | x)$ be as above. Let $\lambda \in \mathcal{L}(R)$, and let W be the Radon-Nikodym derivative from (2.3). Then*

$$\int_X P(\cdot | x) d\lambda(x) = W(\cdot) d\lambda(\cdot). \quad (2.8)$$

Proof. Immediate from the definition. Indeed, for all Borel subset $E \subset X$, the following are equivalent ($f = \chi_E$):

$$\begin{aligned} \int Rf d\lambda &= \int fW d\lambda \\ &\Downarrow \\ \int f(y) P(dy | x) d\lambda(x) &= \int f(x) W(x) d\lambda(x) \\ &\Downarrow \\ \int P(E | x) d\lambda(x) &= \int_E W(y) d\lambda(y) \\ &\Downarrow \\ \int_X P(\cdot | x) d\lambda(x) &= W(\cdot) d\lambda(\cdot) \end{aligned}$$

□

Remark 2.8. In general, $\lambda \neq P(\cdot | x_0)$, $x_0 \in X$. Note that $\lambda = P(\cdot | x_0) \in \mathcal{L}(R)$ iff

$$\begin{aligned} \int_y P(\cdot | y) P(dy | x_0) &= W(\cdot) P(\cdot | x_0), \text{ i.e.,} \\ \int_y P(dz | y) P(dy | x_0) &= W(z) P(dz | x_0) \end{aligned} \quad (2.9)$$

However, condition (2.9) is very restrictive, and it is not satisfied in many cases. See Example 2.9 below.

Example 2.9 (Iterated Function System (IFS); see e.g., [Jor12, DJ09]). Let $X = [0, 1] = \mathbb{R}/\mathbb{Z}$, and $\lambda = \text{Lebesgue measure}$. Fix $v > 0$, a positive function on $[0, 1]$, and set

$$(Rf)(x) = v\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) + v\left(\frac{x+1}{2}\right) f\left(\frac{x+1}{2}\right).$$

Then

$$P(\cdot | x) = \underbrace{v\left(\frac{x}{2}\right) \delta_{\frac{x}{2}} + v\left(\frac{x+1}{2}\right) \delta_{\frac{x+1}{2}}}_{\text{atomic}} \not\ll \underbrace{\lambda}_{\text{non-atomic}}.$$

Assumption (Additional axiom on R). Let R be the positive mapping in Definition 2.1. Assume there exists $\sigma : X \rightarrow X$, measurable and onto, such that

$$R((f \circ \sigma)g) = fRg, \quad \forall f, g \in C(X). \quad (2.10)$$

R in (2.10) is a generalized conditional expectation.

Lemma 2.10. Let R satisfy (2.10) and let $\{P(\cdot | x)\}_{x \in X}$ be the family of conditional measures in Definition 2.3. Then,

$$P(E | x) = \int_E \frac{f(\sigma(y))}{f(x)} P(dy | x) \quad (2.11)$$

for all $f \in C(X)$, and all $E \in \mathcal{B}(X)$; where $\mathcal{B}(X)$ denotes all Borel subsets of X .

Proof. We have

$$\begin{aligned} R((f \circ \sigma)g)(x) &= f(x) R(g)(x) \\ &\Downarrow \\ \int f(\sigma(y)) g(y) P(dy | x) &= f(x) \int g(y) P(dy | x), \quad \forall f, g \in C(X), \forall x \in X. \\ &\Downarrow \quad g = \chi_E \\ \int_E f(\sigma(y)) P(dy | x) &= f(x) P(E | x), \quad \forall f \in C(X), \forall E \in \mathcal{B}(X), \end{aligned}$$

and the assertion follows. \square

Lemma 2.11. Suppose (2.10) holds and $\lambda \in \mathcal{L}(R)$. Set $W = \text{the Radon-Nikodym derivative}$, then the operator $S : f \rightarrow Wf \circ \sigma$ is well-defined and linear in $L^2(\lambda)$ with $C(X)$ as dense domain. In general S is unbounded. Moreover,

$$S \subset R^*, \text{ containment of unbounded operators}, \quad (2.12)$$

where R^* denotes the adjoint operator to R , i.e.,

$$\int_X (Wf \circ \sigma) g d\lambda = \int_X f R(g) d\lambda; \quad (2.13)$$

holds for all $f, g \in C(X)$. That is,

$$R^* f = Wf \circ \sigma, \quad \forall f \in C(X), \quad (2.14)$$

as a weighted composition operator.

Further, the selfadjoint operator RR^* is the multiplication operator:

$$RR^* f = R(W) f, \quad \forall f \in C(X); \quad (2.15)$$

i.e., multiplication by the function $R(W)$.

Proof. For all $f, g \in C(X)$, we have

$$\begin{aligned} \int_X f R(g) d\lambda &\stackrel{\text{by (2.10)}}{=} \int_X R((f \circ \sigma) g) d\lambda \\ &\stackrel{\text{by (2.4)}}{=} \int_X \underbrace{(W(f \circ \sigma))g}_{=S(f)} d\lambda, \end{aligned}$$

and so (2.12)-(2.13) follow. Also,

$$RR^* f \stackrel{\text{by (2.12)}}{=} R(W(f \circ \sigma)) \stackrel{\text{by (2.10)}}{=} f R(W) = mf,$$

where $m = R(W)$. The assertion (2.15) follows from this. \square

Corollary 2.12. S is isometric in $L^2(\lambda) \iff R(W) = 1$.

Corollary 2.13. R defines a bounded operator on $L^2(\lambda)$, i.e., $L^2(\lambda) \xrightarrow{R} L^2(\lambda)$ is bounded $\iff R(W) \in L^\infty(\lambda)$.

Proof. Immediate from (2.15) since

$$\|RR^*\|_{2 \rightarrow 2} = \|R\|_{2 \rightarrow 2}^2 = \|R^*\|_{2 \rightarrow 2}^2. \quad (2.16)$$

\square

Remark 2.14. Let $\lambda \in \mathcal{L}(R)$, $\mu = \lambda \cdot R$, and $W = d\mu/d\lambda$ as before. Even if $W \in L^1(\lambda)$, the following two operators are still well-defined:

$$\begin{aligned} L^2(\lambda) \supset \left\{ \begin{array}{l} C(X) \ni f \xrightarrow{R} Rf \in L^\infty(\lambda) \subset L^2(\lambda) \\ C(X) \ni f \xrightarrow{S} W(f \circ \sigma) \in L^2(\lambda) \end{array} \right\}, \text{ and} \\ \langle Sf, g \rangle_{L^2} = \langle f, Rg \rangle_{L^2}, \quad \forall f, g \in C(X). \end{aligned}$$

Corollary 2.15. Assume $\frac{1}{W}$ is well-defined. Then $\lambda \circ \sigma^{-1} \ll \lambda$, and

$$\frac{d\lambda \circ \sigma^{-1}}{d\lambda} = R\left(\frac{1}{W}\right),$$

where $R\left(\frac{1}{W}\right)$ is defined as in (2.7) of Corollary 2.5.

Proof. Recall the pull-back measure $\lambda \circ \sigma^{-1}$, where $\sigma^{-1}(E) = \{z \in X \mid \sigma(z) \in E\}$, for all Borel sets $E \subset X$. One checks that

$$\int f d\lambda \circ \sigma^{-1} = \int f \circ \sigma d\lambda = \int \frac{1}{W} W f \circ \sigma d\lambda = \int R\left(\frac{1}{W}\right) f d\lambda, \quad \forall f \in C(X);$$

and the assertion follows. \square

Corollary 2.16. *Let R, λ, W be as above, and assume that $\|R(W)\|_\infty \leq 1$. Let h be a function on X solving the equation*

$$Rh = h, \quad h \in L^2(\lambda), \quad (R\text{-harmonic}) \quad (2.17)$$

then the following implication holds:

$$h(x) \neq 0 \implies R(W)(x) = 1. \quad (2.18)$$

Proof. By (2.16), R is contractive, i.e., $\|R\|_{2 \rightarrow 2} = \|R^*\|_{2 \rightarrow 2} \leq 1$, $\|Rf\|_{L^2(\lambda)} \leq \|f\|_{L^2(\lambda)}$; and so $R^*h = h$; and, by (2.15),

$$h = h R(W), \quad \text{pointwise}, \quad (2.19)$$

i.e., $h(x) = h(x) R(W)(x)$, for all $x \in X$, and (2.18) follows. \square

Corollary 2.17. *Suppose $\lambda \in \mathcal{L}(R)$ with $R, \sigma, W = d\mu/d\lambda$ satisfying the usual axioms, then λ is σ -invariant, i.e.,*

$$\int f \circ \sigma d\lambda = \int f d\lambda, \quad \forall f \in C(X) \quad (2.20)$$

\Updownarrow

$$\frac{1}{W} \text{ exists, and } R\left(\frac{1}{W}\right) = 1 \text{ on the support of } \lambda. \quad (2.21)$$

Proof. (2.20) \implies (2.21) follows from Corollary 2.15. (Also see Corollary 2.5.)

Assume (2.21), then

$$\text{LHS}_{(2.20)} = \int \frac{1}{W} \underbrace{W f \circ \sigma}_{R^* f} d\lambda = \int R\left(\frac{1}{W}\right) f d\lambda = \int f d\lambda, \quad \forall f \in C(X).$$

\square

Corollary 2.18. *Let X, R, λ, W, σ be as specified above. Recall that $f \geq 0 \implies Rf \geq 0$, and $R((f \circ \sigma)g) = fR(g)$, $\forall f, g \in C(X)$. Assume further that $W \in L^2(\lambda)$, then*

$$\int_X |W|^2 f \circ \sigma d\lambda = 0, \text{ for some } f \in C(X) \quad (2.22)$$

\Updownarrow

$$\int_X f R(W) d\lambda = 0. \quad (2.23)$$

Proof. Use that $L^2(\lambda) \ni f \xrightarrow{R^*} W f \circ \sigma \in L^2(\lambda)$, we conclude that

$$\begin{aligned} \int_X |W|^2 f \circ \sigma d\lambda &= \int_X W f \circ \sigma \overline{W} d\lambda = \langle R^* f, W \rangle_{L^2(\lambda)} \\ &= \langle f, R(W) \rangle_{L^2(\lambda)} = \int_X f(x) R(\overline{W})(x) d\lambda(x). \end{aligned} \quad (2.24)$$

□

Corollary 2.19. *Let X, R, λ, W, σ be as above, and let $E \subset X$ be a Borel set; then*

$$\int_{\sigma^{-1}(E)} |W|^2 d\lambda = \int_E R(W) d\lambda, \quad (2.25)$$

and so in particular, $R(W) \geq 0$ a.e. on X w.r.t. λ .

Proof. Approximate χ_E with $f \in C(X)$ and use (2.24), we have

$$\int |W|^2 f \circ \sigma d\lambda = \int R(W) f d\lambda,$$

which is (2.25). □

Example 2.20. Let $X = [0, 1] = \mathbb{R}/\mathbb{Z}$, and $d\lambda = dx =$ Lebesgue measure. Fix $W > 0$, a positive function over $[0, 1]$, and set

$$(Rh)(x) = \frac{1}{2} \left(W\left(\frac{x}{2}\right) h\left(\frac{x}{2}\right) + W\left(\frac{x+1}{2}\right) h\left(\frac{x+1}{2}\right) \right). \quad (2.26)$$

Let $\sigma(x) = 2x \bmod 1$, $x \in X$, then

$$\int_0^1 g(x) (Rh)(x) dx = \int_0^1 W(x) g(\sigma(x)) h(x) dx, \quad \forall f, g \in C(X). \quad (2.27)$$

Proof. We introduce the mappings τ_0 and τ_1 , as in Fig 2.1-2.2, so that $\sigma(\tau_i(x)) = x$, for all $x \in X$, $i = 0, 1$. One checks that

$$R((g \circ \sigma)h)(x) = g(x) (Rh)(x). \quad (2.28)$$

Note that $\lambda \in \mathcal{L}(R)$. Indeed, we have

$$\begin{aligned} \lambda(Rh) &= \int_0^1 (Rh)(x) dx \\ &\stackrel{\text{by (2.26)}}{=} \int_0^1 \frac{1}{2} \left((Wh)\left(\frac{x}{2}\right) + (Wh)\left(\frac{x+1}{2}\right) \right) dx \\ &= \int_0^1 (Wh)(x) dx = \int_0^1 h(x) W(x) dx, \end{aligned}$$

and so $\frac{d\mu}{d\lambda}(x) = W(x)$, where $\mu = \lambda \cdot R$. □

Example 2.21. Let R be as in (2.26), and let h be an R -harmonic function, i.e.,

$$(Rh)(x) = \frac{1}{2} \left((Wh)\left(\frac{x}{2}\right) + (Wh)\left(\frac{x+1}{2}\right) \right) = h(x), \quad x \in X = [0, 1]. \quad (2.29)$$

Setting $\hat{h}(n) = \int_0^1 e(nx) h(x) dx$, with $e(nx) := e^{i2\pi nx}$, it follows from (2.29) that

$$\begin{aligned} \hat{h}(n) &= \int_0^1 e(nx) (Rh)(x) dx \\ &= \int_0^1 W(x) e(2nx) h(x) dx = (Wh)^\wedge(2n), \quad \forall n \in \mathbb{Z}. \end{aligned}$$

An iteration gives

$$\begin{aligned}
 \widehat{h}(n) &= \int_0^1 W(x) e(2nx) (Rh)(x) dx \\
 &= \int_0^1 W(x) W(2x) e(2^2 nx) h(x) dx \\
 &\dots \\
 &= \int_0^1 W(x) W(2x) \dots W(2^{k-1}x) e(2^k nx) h(x) dx,
 \end{aligned}$$

and so

$$\widehat{h}(n) = (W_k h)^\wedge(2^k n), \quad \forall n \in \mathbb{Z}, \forall k = 0, 1, 2, \dots;$$

where $W_k(x) := W(x) W(2x) \dots W(2^{k-1}x)$.

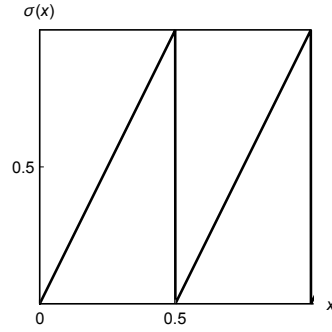


FIGURE 2.1. $\sigma(x) = 2x \bmod 1$

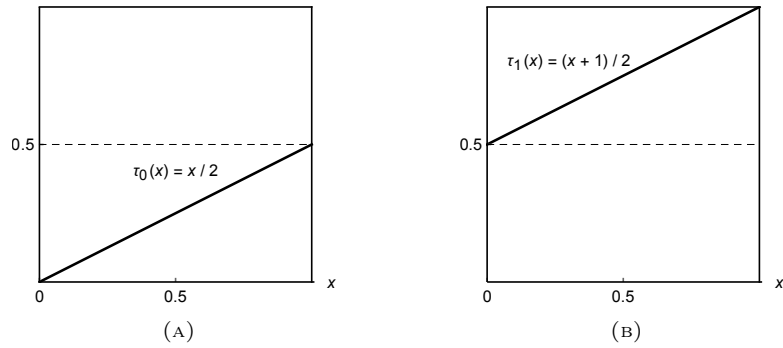


FIGURE 2.2. $\tau_0 = \frac{x}{2}, \tau_1 = \frac{x+1}{2}$

3. ITERATED FUNCTION SYSTEMS: THE GENERAL CASE

In this section we discuss a subfamily of systems where the generalized transfer operator R is associated with an Iterated Function system (IFS).

Let X be a compact Hausdorff space, $n \in \mathbb{N}$, and let

$$\tau_i : X \longrightarrow X, \quad 1 \leq i \leq n \quad (3.1)$$

be a system of endomorphisms. Let

$$p_i > 0, \text{ s.t. } \sum_{i=1}^n p_i = 1. \quad (3.2)$$

Following [Hut81, Jor12, FH09, Urb09, DABJ09, DJ09], we say that (3.1)-(3.2) is an *Iterated Function System* (IFS) if there is a Borel probability measure λ on X such that

$$\sum_{i=1}^n p_i \int_X f(\tau_i(x)) d\lambda(x) = \int_X f(x) d\lambda(x) \quad (3.3)$$

holds for all $f \in C(X)$. Note that (3.3) may also be expressed as follows:

$$\sum_i p_i \lambda \circ \tau_i^{-1} = \lambda. \quad (3.4)$$

The measure λ is called an IFS measure.

Let $W \in L^1(\lambda)$, $W \geq 0$, and set

$$(R_W f)(x) = \sum_{i=1}^n p_i (W f)(\tau_i(x)), \quad x \in X, f \in C(X), \quad (3.5)$$

where $(W f)(\tau_i(x)) := W(\tau_i(x)) f(\tau_i(x))$.

Lemma 3.1. *If W is as above, and if λ is an IFS measure, then $\lambda \in \mathcal{L}_1(R_W)$, see Remark 2.6.*

Proof. We establish the conclusion by verifying that, under the assumptions, we have

$$\int_X (R_W f)(x) d\lambda(x) = \int_X W(x) f(x) d\lambda(x), \quad \forall f \in C(X), \quad (3.6)$$

i.e., W is the Radon-Nikodym derivative, $d\mu_W/d\lambda = W$, where $\mu_W = \lambda \cdot R_W$. Indeed,

$$\begin{aligned} \text{LHS}_{(3.6)} & \stackrel{\text{by (3.5)}}{=} \sum_{i=1}^n p_i \int_X (W f)(\tau_i(x)) d\lambda(x) \\ & \stackrel{\text{by (3.3)}}{=} \int_X (W f)(x) d\lambda(x) = \text{RHS}_{(3.6)}. \end{aligned}$$

□

Remark 3.2. The setting of Example 2.21, we have an IFS corresponding to the two mappings in Figure 2.2, and, in this setting, the corresponding IFS measure λ on the unit interval $X = [0, 1]$ can then easily be checked to be the restriction to $[0, 1]$ of the standard Lebesgue measure. It is important to mention that there is a rich literature on IFS measures, see e.g., [Hut81, Jor12, FH09, Urb09, DABJ09, DJ09], and the variety of IFS measures associated to function systems includes explicit classes measures of fractal dimension.

4. THE SET $\mathcal{L}_1(R)$ FROM A QUADRATIC ESTIMATE

In order to build a path-space probability space from a given generalized transfer operator R , a certain spectral property for R must be satisfied, and we discuss this below; see Theorem 4.1. The statement of the problem requires the introduction of a Hilbert space of sigma functions, also called square densities.

Let X be a locally compact Hausdorff space, and let $R : C(X) \rightarrow \mathcal{M}(X)$ be given, subject to the conditions in Definition 2.1 and Remark 2.6.

For every probability measure λ on X , we apply the Radon-Nikodym decomposition (see [Rud87]) to the measure λR , getting

$$\lambda R = \mu_{abs} + \mu_{sing} \quad (4.1)$$

where the two terms on the RHS in (4.1) are absolutely continuous w.r.t λ , respectively, with μ_{sing} and λ mutually singular. Hence there is a positive $W_\lambda \in \mathcal{L}_1(\lambda)$ such that $\mu_{abs} = W_\lambda d\lambda$. When λ is fixed, set

$$\tau = \frac{1}{2} (\mu_{abs} + \lambda R). \quad (4.2)$$

We have the following:

Theorem 4.1. *Let (X, R) be as described above, and let $Prob(X)$ be the convex set of all probability measures on X . For $\lambda \in Prob(X)$, let τ be the corresponding measure given by (4.2). Then $\mathcal{L}_1(R) \neq 0$ if and only if*

$$\inf_{\lambda \in Prob(X)} \int_X \left| \sqrt{\frac{d(\lambda R)}{d\tau}} - W_\lambda \sqrt{\frac{d\lambda}{d\tau}} \right|^2 d\tau = 0. \quad (4.3)$$

Proof. To carry out the proof details, we shall make use of the Hilbert space $Sig(X)$ of sigma-functions on X . While it has been used in, for example [Nel69, KM46, Hid80, Jor11], we shall introduce the basic facts which will be needed.

Elements in $Sig(X)$ are equivalence classes of pairs (f, μ) , where $f \in L^2(\mu)$, and μ is a positive finite measure on X ; we say that $(f, \mu) \sim (g, \nu)$ for two such pairs iff $\tau = \frac{1}{2}(\mu + \nu)$ satisfies

$$f \sqrt{\frac{d\mu}{d\tau}} = g \sqrt{\frac{d\nu}{d\tau}} \quad \text{a.e. on } X \text{ w.r.t. } \tau. \quad (4.4)$$

If $class(f_i, \mu_i)$, $i = 1, 2$, are two equivalence classes, then the operations in $Sig(X)$ are as follows: First set $\tau_s = \frac{1}{2}(\mu_1 + \mu_2)$, then the inner product in $Sig(X)$ is

$$\int_X \bar{f}_1 \sqrt{\frac{d\mu_1}{d\tau_s}} f_2 \sqrt{\frac{d\mu_2}{d\tau_s}} d\tau_s,$$

and the sum is

$$class \left(f_1 \sqrt{\frac{d\mu_1}{d\tau_s}} + f_2 \sqrt{\frac{d\mu_2}{d\tau_s}}, \tau_s \right).$$

It is known that these definitions pass to equivalence classes; and that $Sig(X)$ is a Hilbert space; in particular, it is complete.

In order to complete the proof of the theorem, we shall need the following facts about the Hilbert space $Sig(X)$; see e.g., [Nel69]: First some notation; we set

$$f \sqrt{d\mu} = class(f, \mu) \in Sig(X); \quad (4.5)$$

and when μ is fixed, we set $\mathfrak{M}_2(\mu)$ to be the closed subspace in $Sig(X)$ spanned by

$$\left\{ f\sqrt{d\mu} \mid f \in L^2(\mu) \right\}.$$

We then have:

$$\left\| f\sqrt{d\mu} \right\|_{Sig(X)}^2 = \|f\|_{L^2(\mu)}^2 = \int_X |f|^2 d\mu; \quad (4.6)$$

and so, in particular,

$$L^2(\mu) \ni f \longmapsto f\sqrt{d\mu} \in Sig(X) \quad (4.7)$$

defines an isometry with range $\mathfrak{M}_2(\mu)$. We shall abbreviate $\sqrt{d\mu}$ as $\sqrt{\mu}$. For two measures μ and ν , the following three facts holds:

$$\begin{aligned} [\mu \ll \nu] &\iff \mathfrak{M}_2(\mu) \subseteq \mathfrak{M}_2(\nu), \\ [\mu \approx \nu] &\iff \mathfrak{M}_2(\mu) = \mathfrak{M}_2(\nu), \text{ and} \\ \left[\begin{array}{c} \mu \text{ and } \nu \text{ are} \\ \text{mutually singular} \end{array} \right] &\iff \mathfrak{M}_2(\mu) \perp \mathfrak{M}_2(\nu). \end{aligned} \quad (4.8)$$

As a result, we note that therefore, the decomposition in (4.1) is orthogonal in $Sig(X)$, and further that a fixed $\lambda \in Prob(X)$ is in $\mathcal{L}_1(R)$ if and only if

$$\mathfrak{M}_2(\lambda R) \subseteq \mathfrak{M}_2(\lambda) \quad (4.9)$$

$$\Updownarrow$$

$$\inf_{\lambda \in Prob(X)} \left\| \sqrt{\lambda R} - W_\lambda \sqrt{\lambda} \right\|_{Sig(X)}^2 = 0 \quad (4.10)$$

Moreover, (4.9) is a restatement of (4.3).

We now turn to the conclusions in the theorem: One implication is clear. If now the infimum in (4.10) is zero, then there is a sequence $\{\lambda_n\} \subset Prob(X)$ such that

$$\lim_n \left\| \sqrt{\lambda_n R} - W_{\lambda_n} \sqrt{\lambda_n} \right\|_{Sig(X)}^2 = 0. \quad (4.11)$$

Combining (4.10) and (4.11), and possibly passing to a subsequence, we conclude that there is a sequence $W_{\lambda_n} \sqrt{\lambda_n}$ which is convergent in $Sig(X)$. Let the limit be $W_{\lambda_0} \sqrt{\lambda_0}$, and it follows that $\lambda_0 \in \mathcal{L}_1(R)$. \square

Remark 4.2. Since $Sig(X)$ is a Hilbert space, we conclude that the sequence $\{W_{\lambda_n} \sqrt{\lambda_n}\}_n$ in $Sig(X)$ satisfies

$$\lim_n \left\| \sqrt{\lambda_0 R} - W_{\lambda_n} \sqrt{\lambda_n} \right\|_{Sig(X)}^2 = 0;$$

where the desired measure $\lambda_0 \in \mathcal{L}_1(R)$ may be taken to be

$$d\lambda_0(\cdot) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d\lambda_n(\cdot)}{\lambda_n(X)}.$$

5. FROM ENDOMORPHISM TO AUTOMORPHISM

In this section (Theorem 5.2), we build a path-space probability space from a given generalized transfer operator R assumed to satisfy the spectral property from above.

There is a generalized family of multi-resolution measures on *solenoids*, and we shall need the following facts (see e.g., [Hut81, Jor12, FH09, Urb09, DABJ09, DJ09]):

Let X be a compact Hausdorff space, and let $\sigma : X \rightarrow X$ be a continuous endomorphism onto X . Let

$$\Omega := \prod_0^\infty X = X \times X \times \cdots \quad (5.1)$$

be the infinite Cartesian product with coordinate mappings $Z_n : \Omega \rightarrow X$,

$$Z_n(x_0, x_1, x_2, \dots) = x_n \in X, \quad n \in 0, 1, 2, \dots \quad (5.2)$$

The associated *solenoid* $Sol_\sigma(X) \subset \prod_0^\infty X$ is defined as follows:

$$Sol_\sigma(X) = \{(x_n)_0^\infty \in \Omega \mid \sigma(x_{n+1}) = x_n, n = 0, 1, 2, \dots\}; \quad (5.3)$$

and set

$$\tilde{\sigma}(x_0, x_1, x_2, \dots) := (\sigma(x_0), x_0, x_1, x_2, \dots). \quad (5.4)$$

We give $Sol_\sigma(X)$ its relative projective topology, and note that the restricted random variable $(Z_n)_{n=0}^\infty$ from (5.2) are then continuous. Moreover $\tilde{\sigma}$, in (5.4), is invertible with

$$\tilde{\sigma}^{-1}(x_0, x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, \dots), \quad (5.5)$$

$$\tilde{\sigma}\tilde{\sigma}^{-1} = \tilde{\sigma}^{-1}\tilde{\sigma} = Id_{Sol_\sigma(X)}. \quad (5.6)$$

Let (X, R, λ, W) be as specified in Section 2. In particular, R is positive, i.e., $f \in C(X)$, $f \geq 0 \implies R(f) \geq 0$, and

$$R((f \circ \sigma)g) = fR(g), \quad \forall f, g \in C(X). \quad (5.7)$$

Moreover, W is the Radon-Nikodym derivative of the measure $f \rightarrow \lambda(R(f))$ w.r.t. λ , i.e.,

$$\int_X R(f) d\lambda = \int_X fW d\lambda, \quad \forall f \in C(X). \quad (5.8)$$

Let $h \in L^\infty(\lambda)$, $h \geq 0$, satisfying

$$Rh = h, \text{ and } \int_X h d\lambda = 1. \quad (5.9)$$

Remark 5.1. In view of equations (2.8) and (5.9), it is natural to think of these conditions as a generalized Perron-Frobenius property for R .

Theorem 5.2. *With the assumptions (5.7)-(5.9), we have the following conclusions:*

- (1) *For every $x \in X$, there is a unique Borel probability measure \mathbb{P}_x on $Sol_\sigma(X)$ such that for all n , and all $f_0, f_1, \dots, f_n \in C(X)$,*

$$\begin{aligned} & \int_{Z_0^{-1}(x)} (f_0 \circ Z_0)(f_1 \circ Z_1) \cdots (f_n \circ Z_n) d\mathbb{P}_x \\ &= f_0(x) R(f_1 R(f_2 R(\cdots R(f_n h) \cdots))) (x). \end{aligned} \quad (5.10)$$

(2) Moreover, setting

$$\mathbb{P} = \int_X \mathbb{P}_x d\lambda(x), \quad (5.11)$$

we get that \mathbb{P} is a probability measure on $\text{Sol}_\sigma(X)$ such that

$$\mathbb{E}_{\mathbb{P}}(\cdots | Z_0 = x) = \mathbb{P}_x \quad (5.12)$$

where the LHS in (5.12) is the conditional measure, and the RHS is the measure from (5.10).

(3) We have the following Radon-Nikodym derivative:

$$\frac{d\mathbb{P} \circ \tilde{\sigma}}{d\mathbb{P}} = W \circ Z_0, \quad (5.13)$$

as an identity of the two functions specified in (5.13). Equivalently, setting

$$U\psi = \left(\sqrt{W \circ Z_0}\right) \psi \circ \tilde{\sigma}, \quad \psi \in L^2(\text{Sol}_\sigma(X), \mathbb{P}),$$

then U is a unitary operator in $L^2(\text{Sol}_\sigma(X), \mathbb{P})$.

Proof. This is the basic Kolmogorov inductive limit construction. We note that, by Stone-Weierstrass, the space of cylinder-functions

$$(f_0 \circ Z_0)(f_1 \circ Z_1) \cdots (f_n \circ Z_n) \quad (5.14)$$

is dense in $C(\text{Sol}_\sigma(X))$. Fix $x \in X$, and start with $Z_0^{-1}(x)$, set

$$L_n^x(f_1, f_2, \cdots, f_n) = R(f_1 R(f_2 \cdots R(f_n h) \cdots))(x). \quad (5.15)$$

We get the desired consistency:

$$L_{n+1}^x(f_1, f_2, \cdots, f_n, \mathbb{1}) = L_n^x(f_1, f_2, \cdots, f_n) \quad (5.16)$$

where $\mathbb{1}$ denotes the constant function 1 on X . Indeed,

$$\begin{aligned} R(f_{n-1} R(f_n R(\mathbb{1}h)))(x) &= R(f_{n-1} R(f_n R(h)))(x) \\ &= R(f_{n-1} R(f_n h))(x), \quad (\text{by (5.9)}) \end{aligned}$$

as claimed in (5.16). \square

Lemma 5.3. Let $R, X, P(\cdot | x)$ be as above. Assume $h \geq 0$ on X , and $Rh = h$. Then

$$|R(fh)(x)| \leq \|f\|_\infty h(x) \quad (5.17)$$

where $\|f\|_\infty$ is the $P(\cdot | x)$ L^∞ -norm on functions on X .

Proof. We may apply Cauchy-Schwarz to $P(\cdot | x)$ in a sequence of steps as follows:

$$\begin{aligned} |R(fh)(x)| &= \left| R(fh^{\frac{1}{2}}h^{\frac{1}{2}})(x) \right| & (5.18) \\ &\leq (R(|f|^2 h)(x))^{\frac{1}{2}} (R(h)(x))^{\frac{1}{2}} & \text{by Schwarz} \\ &= (R(|f|^2 h)(x))^{\frac{1}{2}} h(x)^{\frac{1}{2}} & \text{since } Rh = h \\ &\leq \underbrace{R(|f|^{2p} h)(x)^{\frac{1}{2p}}}_{\longrightarrow \|f\|_\infty} \underbrace{h(x)^{\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^p}}}_{\longrightarrow h(x)} & \text{by induction, and let } p \longrightarrow \infty \end{aligned}$$

An elementary result in measure theory (see [Rud87]) shows that

$$\lim_{p \rightarrow \infty} R(|f|^{2^p} h)(x)^{\frac{1}{2^p}} = \|f\|_{\infty}; \quad (5.19)$$

and so the desired estimate (5.17) holds. \square

Corollary 5.4. *Let $R, h, \sigma, W, \lambda \in \mathcal{L}(R)$ be as above, where $\mu = \lambda \cdot R$, and $W = \frac{d\mu}{d\lambda}$. Assume $h > 0$ on X , and $Rh = h$. Let \mathbb{P} and \mathbb{P}_x be the measures on $Sol_{\sigma}(X)$ as in Theorem 5.2, where \mathbb{P}_x is determined by*

$$\begin{aligned} \int_{Z_0^{-1}(x)} (f_1 \circ Z_1) \cdots (f_n \circ Z_n) d\mathbb{P}_x &= L_n^x(f_1, \dots, f_n) \\ &= R(f_1 R(f_2 \cdots R(f_n h) \cdots))(x). \end{aligned} \quad (5.20)$$

Then,

$$\left| \int \psi d\mathbb{P}_x \right| \leq \|\psi\|_{\infty} h(x), \quad \forall \psi. \quad (5.21)$$

That is,

$$\left| \frac{\mathbb{E}(\psi | x)}{h(x)} \right| \leq \|\psi\|_{\infty}, \quad \forall \psi; \quad (5.22)$$

in particular, $\mathbb{E}(\psi | x) = \int \psi d\mathbb{P}_x$ (= conditional expectation) is well-defined. See (5.12).

Corollary 5.5. *Let X, R, σ, λ, h be as described above; in particular, $Rh = h$ is assumed. Let $\{\mathbb{P}_x\}_{x \in X}$ be the measures from Corollary 5.4. Then*

$$h(x) = \mathbb{P}_x(Z_0^{-1}(x)) \text{ for all } x \in X.$$

Corollary 5.6. *Let R, h, σ, W, λ be as above, and let \mathbb{P} and \mathbb{P}_x be the corresponding measures on $Sol_{\sigma}(X)$, then V_0 is isometric, where*

$$V_0 : L^2(X, h d\lambda) \longrightarrow L^2(Sol_{\sigma}(X), \mathbb{P})$$

is given by

$$V_0 g = g \circ Z_0, \quad g \in L^2(X, h d\lambda), \quad (5.23)$$

and

$$(V^* \psi)(x) = \frac{\mathbb{E}(\psi | x)}{h(x)}, \quad \forall x \in X, \forall \psi \in L^2(Sol_{\sigma}(X), \mathbb{P}). \quad (5.24)$$

Proof. Since $\mathbb{P} = \int_X \mathbb{P}_x d\lambda(x)$, it follows that V_0 in (5.23) is isometric, i.e.,

$$\|V_0 g\|_{L^2(\mathbb{P})}^2 = \int_X |g|^2 h d\lambda = \|g\|_{L^2(h d\lambda)}^2, \quad \forall g \in C(X).$$

To prove (5.24), we must establish

$$\int_{Sol_{\sigma}(X)} (g \circ Z_0) \psi d\mathbb{P} = \int_X g(x) \mathbb{E}(\psi | x) d\lambda(x). \quad (5.25)$$

Since the space of the cylinder functions $\psi = (f_0 \circ Z_0)(f_1 \circ Z_1) \cdots (f_n \circ Z_n)$ is dense in $C(Sol_{\sigma}(X))$, it suffices to prove (5.25) for ψ . But then

$$(g \circ Z_0) \psi = (g f_0) \circ Z_0 (f_1 \circ Z_1) \cdots (f_n \circ Z_n),$$

and so (5.25) follows from (5.20). \square

Corollary 5.7. Fix $x \in X$, and set

$$\mathbb{E}(\psi \mid x) = \int_{Z_0^{-1}(x)} \psi d\mathbb{P}_x, \quad \psi \in L^2(\mathbb{P}). \quad (5.26)$$

For all $n \in \mathbb{N}$, if $A_i \subset X$, $i = 1, \dots, n$, are Borel sets, then

$$\begin{aligned} & \mathbb{P}_x(Z_1 \in A_1, Z_2 \in A_2, \dots, Z_n \in A_n) \\ &= \int_{A_1} \int_{A_2} \dots \int_{A_n} h(y_n) P(dy_n \mid y_{n-1}) \dots P(dy_2 \mid y_1) P(dy_1 \mid x). \end{aligned} \quad (5.27)$$

Proof. Recall that $\chi_A \circ Z_i = \chi_{Z_i^{-1}(A)}$, if $A \subset X$ is a Borel set; and

$$Z_i^{-1}(A) = \{x \in \text{Sol}_\sigma(X) \mid Z_i(x) \in A\}, \quad (5.28)$$

where $Z_i(x_0, x_1, x_2, \dots) = x_i$ is the coordinate mapping. Also, $P(\cdot \mid x)$ satisfies

$$R(f)(x) = \int_X f(y) P(dy \mid x), \quad \forall x \in X. \quad (5.29)$$

Now set $f_i = \chi_{A_i}$, with $A_i \subset X$ Borel sets, and apply the mapping

$$f_i \longrightarrow R(f_1 R(f_2 \dots R(f_n h) \dots))(x).$$

If we specialize (5.27) to individual transition probabilities, we get, $x \in X$, $A \subset X$ a Borel set, and

$$\begin{aligned} \mathbb{P}(Z_1 \in A \mid Z_0 = x) &= \int_A h(y) P(dy \mid x); \\ \mathbb{P}(Z_2 \in B, Z_1 \in A \mid Z_0 = x) &= \int_A \int_B h(y_2) P(dy_2 \mid y_1) P(dy_1 \mid x), \quad y_1 \in A, y_2 \in B. \end{aligned}$$

Note that, fix $n > 1$, then

$$\begin{aligned} \mathbb{P}(Z_n \in A \mid Z_0 = x) &= \mathbb{P}_x(Z_n \in A) = R^n(\chi_A h)(x), \text{ and} \\ \mathbb{P}_x(Z_{n+1} \in B, Z_n \in A) &= R^n(\chi_A R(\chi_B h))(x) \neq \mathbb{P}_x(Z_2 \in B, Z_1 \in A), \end{aligned}$$

so it is not Markov. \square

Hence the transition from n to $n+1$ gets more “flat” as n increases, the transition probability evens out with time.

5.1. Multi-Resolutions. Let X, σ, R, h , and λ be as in the setting of Theorem 5.2 above. In particular, we are assuming that:

- (i) $R((f \circ \sigma)g) = fR(g)$, $\forall f, g \in C(X)$,
- (ii) $Rh = h$, $h \geq 0$,
- (iii) $\int R(f) d\lambda = \int_X f W d\lambda$, $\forall f \in C(X)$, and
- (iv) $\int_X h(x) d\lambda(x) = 1$.

We then pass to the probability space $(\text{Sol}_\sigma(X), \mathbb{P}_x, \mathbb{P})$ from the conclusion in Theorem 5.2.

Definition 5.8. Let \mathcal{H} be a Hilbert space, and $\{\mathcal{H}_n\}_{n \in \mathbb{N}_0}$ a given system of closed subspaces such that $\mathcal{H}_n \subset \mathcal{H}_{n+1}$, for all n .

We further assume that $\cup_n \mathcal{H}_n$ is dense in \mathcal{H} , and that a unitary operator U in \mathcal{H} satisfying $U(\mathcal{H}_n) \subset \mathcal{H}_{n-1}$, for all $n \in \mathbb{N}$. Then we say that $((\mathcal{H}_n)_{n \in \mathbb{N}_0}, U)$ is a *multi-resolution* for the Hilbert space \mathcal{H} .

Theorem 5.9. *Let $\mathcal{H} = L^2(\text{Sol}_\sigma(X), \mathbb{P})$ be the Hilbert space from the construction in Theorem 5.2, and let \mathcal{H}_n be the closed subspaces defined from the random walk process $(Z_n)_{n \in \mathbb{N}}$. Finally, let U be the operator in part (3) of Theorem 5.2. Then this constitutes a multi-resolution.*

Proof. As indicated above, the setting is specified in Theorem 5.2, and we set

$$\mathcal{H} := L^2(\text{Sol}_\sigma(X), \mathbb{P});$$

and, for each $n \in \mathbb{N}$, let $\mathcal{H}_n \subset \mathcal{H}$, be the closed subspace spanned by

$$\{f \circ Z_n \mid f \in C(X)\}. \quad (5.30)$$

Since

$$f \circ Z_n = (f \circ \sigma) \circ Z_{n+1} \quad (5.31)$$

it follows that $\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$. It further follows from Theorem 5.2 that $\cup_{n \in \mathbb{N}} \mathcal{H}_n$ is dense in \mathcal{H} . And, finally, the unitary operator U from part (3) of Theorem 5.2 satisfies

$$U(\mathcal{H}_n) \subset \mathcal{H}_{n-1}, \quad \forall n \in \mathbb{N}. \quad (5.32)$$

□

Corollary 5.10. *Let $X, \sigma, R, h, \lambda, \mathbb{P}$ be as stated above; and let $((\mathcal{H}_n), U)$ be the corresponding multi-resolution from Theorem 5.9.*

Then $\mathcal{H}_0 \simeq L^2(X, h d\lambda)$, and $\cap_{n \geq 0} U^n \mathcal{H}_m = \mathcal{H}_0$ holds for all $m \in \mathbb{N}$. Finally, U restricts to a unitary operator in $\mathcal{H} \ominus \mathcal{H}_0$; and the spectrum of this restriction is pure Lebesgue spectrum, i.e., there is a Hilbert space \mathcal{K} (the multiplicity space) such that $U|_{\mathcal{H} \ominus \mathcal{H}_0}$ is unitarily equivalent to a subshift of the bilateral shift S in $L^2(\mathbb{T}, \text{Leb}; \mathcal{K})$, where \mathbb{T} is the circle group $\{z \in \mathbb{C} \mid |z| = 1\}$, and the bilateral shift is then given on functions $\psi \in L^2(\mathbb{T}, \text{Leb}; \mathcal{K})$ by

$$(S\psi)(z) = z\psi(z), \quad \psi : \mathbb{T} \longrightarrow \mathcal{K}, \quad z \in \mathbb{T}, \text{ multiplication by } z.$$

Proof. The conclusion follows from an application of the Stone-von Neumann uniqueness theorem [Sum01] combined with the present theorems in Sections 4 and 5 above. (For more details on the spectral representation for operators with multi-resolution, see also [LP67].) □

6. HARMONIC FUNCTIONS FROM FUNCTIONAL MEASURES

Let $R : C(X) \longrightarrow \mathcal{M}(X)$ be as specified in (2.1); and let the measure system $\{P(\cdot \mid x)\}_{x \in X}$ be as specified in (2.3).

Let (Ω, \mathcal{F}) be a measure space; i.e., \mathcal{F} is a specified sigma-algebra of events in a given sample space Ω , and let Z_0 be an X -valued random variable, i.e., it is assumed that $Z^{-1}(A) \in \mathcal{F}$ for every Borel set $A \subset X$. For recent applications, we refer to [AJ12, JP12, JKS12, JPT15, CJ15].

Theorem 6.1. *Let $R, X, \Omega, \mathcal{F}$, and Z_0 be as specified above. Suppose $\{\mathbb{P}_x\}_{x \in X}$ is a system of positive measures on Ω indexed by X , and set*

$$h(x) = \mathbb{P}_x(Z_0^{-1}(x)), \quad x \in X. \quad (6.1)$$

Assume that

$$\int_X \mathbb{P}_y(\cdot) P(dy \mid x) = \mathbb{P}_x(\cdot), \quad (6.2)$$

then h in (6.1) is harmonic for R , i.e., we have

$$R(h) = h, \text{ pointwise on } X. \quad (6.3)$$

Proof. Using (2.5) in Definition 2.3, we get the following:

$$\begin{aligned}
 (Rh)(x) &= \int_X h(y) P(dy | x) \\
 &\stackrel{\text{by (6.1)}}{=} \int_X \mathbb{P}_y(Z_0^{-1}(y)) P(dy | x) \\
 &\stackrel{\text{by (6.2)}}{=} \mathbb{P}_x(Z_0^{-1}(x)) = h(x), \quad x \in X.
 \end{aligned}$$

□

Corollary 6.2. *Let R be a transfer operator. Then if $\lambda \in \mathcal{L}_1(R)$, then there is a solution $h \geq 0$, on X , to $Rh = h$, and $\int_X h(x) d\lambda(x) = 1$.*

Proof. This is a conclusion of Corollary 5.5 and Theorem 6.1. Indeed, given $\lambda \in \mathcal{L}_1(X)$, let $\{\mathbb{P}_x\}_{x \in X}$ be the system from Theorem 5.2, then $h(x) = \mathbb{P}_x(Z_0^{-1}(x))$ is the desired solution. □

Acknowledgement. The co-authors thank the following colleagues for helpful and enlightening discussions: Professors Sergii Bezuglyi, Ilwoo Cho, Paul Muhly, Myung-Sin Song, Wayne Polyzou, and members in the Math Physics seminar at The University of Iowa.

REFERENCES

- [AJ12] Daniel Alpay and Palle E. T. Jorgensen, *Stochastic processes induced by singular operators*, Numer. Funct. Anal. Optim. **33** (2012), no. 7-9, 708–735. MR 2966130
- [Bea91] Alan F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991, Complex analytic dynamical systems. MR 1128089 (92j:30026)
- [BJKR02] Ola Bratteli, Palle E. T. Jorgensen, Ki Hang Kim, and Fred Roush, *Corrigendum to the paper: “Decidability of the isomorphism problem for stationary AF-algebras and the associated ordered simple dimension groups”* [Ergodic Theory Dynam. Systems **21** (2001), no. 6, 1625–1655; MR1869063 (2002h:46088)], Ergodic Theory Dynam. Systems **22** (2002), no. 2, 633. MR 1898809
- [BJO04] Ola Bratteli, Palle E. T. Jorgensen, and Vasyil’ Ostrovs’kyi, *Representation theory and numerical AF-invariants. The representations and centralizers of certain states on \mathcal{O}_d* , Mem. Amer. Math. Soc. **168** (2004), no. 797, xviii+178. MR 2030387 (2005i:46069)
- [CJ15] Ilwoo Cho and Palle E. T. Jorgensen, *Matrices induced by arithmetic functions, primes and groupoid actions of directed graphs*, Spec. Matrices **3** (2015), 123–154. MR 3370362
- [DABJ09] Geoffrey Decrouez, Pierre-Olivier Amblard, Jean-Marc Brossier, and Owen Jones, *Galtson-Watson iterated function systems*, J. Phys. A **42** (2009), no. 9, 095101, 17. MR 2525528 (2010i:28009)
- [Dan01] Alexandre I. Danilenko, *Strong orbit equivalence of locally compact Cantor minimal systems*, Internat. J. Math. **12** (2001), no. 1, 113–123. MR 1812067 (2002j:37016)
- [DJ09] Dorin Ervin Dutkay and Palle E. T. Jorgensen, *Probability and Fourier duality for affine iterated function systems*, Acta Appl. Math. **107** (2009), no. 1-3, 293–311. MR 2520021 (2010g:37011)
- [FH09] De-Jun Feng and Huyi Hu, *Dimension theory of iterated function systems*, Comm. Pure Appl. Math. **62** (2009), no. 11, 1435–1500. MR 2560042 (2010i:37049)
- [GF16] Pablo Guarino and Edson de Faria, *Real bounds and Lyapunov exponents*, Discrete Contin. Dyn. Syst. **36** (2016), no. 4, 1957–1982. MR 3411549
- [Hid80] Takeyuki Hida, *Brownian motion*, Applications of Mathematics, vol. 11, Springer-Verlag, New York-Berlin, 1980, Translated from the Japanese by the author and T. P. Speed. MR 562914 (81a:60089)
- [HPS92] Richard H. Herman, Ian F. Putnam, and Christian F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. **3** (1992), no. 6, 827–864. MR 1194074 (94f:46096)

- [Hut81] John E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747. MR 625600 (82h:49026)
- [JKS12] Palle E. T. Jorgensen, Keri A. Kornelson, and Karen L. Shuman, *An operator-fractal*, Numer. Funct. Anal. Optim. **33** (2012), no. 7-9, 1070–1094. MR 2966145
- [Jor11] Palle E. T. Jorgensen, *Representations of Lie algebras built over Hilbert space*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **14** (2011), no. 3, 419–442. MR 2847247
- [Jor12] ———, *Ergodic scales in fractal measures*, Math. Comp. **81** (2012), no. 278, 941–955. MR 2869044
- [JP12] P. E. T. Jorgensen and A. M. Paolucci, *q-frames and Bessel functions*, Numer. Funct. Anal. Optim. **33** (2012), no. 7-9, 1063–1069. MR 2966144
- [JPT15] Palle Jorgensen, Steen Pedersen, and Feng Tian, *Spectral theory of multiple intervals*, Trans. Amer. Math. Soc. **367** (2015), no. 3, 1671–1735. MR 3286496
- [KLTMV12] Herb Kunze, Davide La Torre, Franklin Mendivil, and Edward R. Vrscay, *Fractal-based methods in analysis*, Springer, New York, 2012. MR 3014680
- [KM46] Shizuo Kakutani and George W. Mackey, *Ring and lattice characterization of complex Hilbert space*, Bull. Amer. Math. Soc. **52** (1946), 727–733. MR 0016534 (8,31e)
- [LP67] P. D. Lax and R. S. Phillips, *Scattering theory for transport phenomena*, Functional Analysis (Proc. Conf., Irvine, Calif., 1966), Academic Press, London; Thompson Book Co., Washington, D.C., 1967, pp. 119–130. MR 0220099 (36 #3166)
- [Mat04] Kengo Matsumoto, *Strong shift equivalence of symbolic dynamical systems and Morita equivalence of C^* -algebras*, Ergodic Theory Dynam. Systems **24** (2004), no. 1, 199–215. MR 2041268 (2004j:37016)
- [Mat06] Hiroki Matui, *Some remarks on topological full groups of Cantor minimal systems*, Internat. J. Math. **17** (2006), no. 2, 231–251. MR 2205435 (2007f:37011)
- [MNB16] Anotida Madzvamuse, Hussaini Ndakwo, and Raquel Barreira, *Stability analysis of reaction-diffusion models on evolving domains: The effects of cross-diffusion*, Discrete Contin. Dyn. Syst. **36** (2016), no. 4, 2133–2170. MR 3411557
- [Nel69] Edward Nelson, *Topics in dynamics. I: Flows*, Mathematical Notes, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1969. MR 0282379 (43 #8091)
- [Pes13] Isaac Z. Pesenson, *Multiresolution analysis on compact Riemannian manifolds*, Multiscale analysis and nonlinear dynamics, Rev. Nonlinear Dyn. Complex., Wiley-VCH, Weinheim, 2013, pp. 65–82. MR 3221687
- [Rud87] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157 (88k:00002)
- [Sum01] Stephen J. Summers, *On the Stone-von Neumann uniqueness theorem and its ramifications*, John von Neumann and the foundations of quantum physics (Budapest, 1999), Vienna Circ. Inst. Yearb., vol. 8, Kluwer Acad. Publ., Dordrecht, 2001, pp. 135–152. MR 2042745
- [TSI⁺15] Satoyuki Tanaka, Shogo Sannomaru, Michiya Imachi, Seiya Hagihara, Shigenobu Okazawa, and Hiroshi Okada, *Analysis of dynamic stress concentration problems employing spline-based wavelet Galerkin method*, Eng. Anal. Bound. Elem. **58** (2015), 129–139. MR 3360371
- [Urb09] Mariusz Urbański, *Geometric rigidity for class S of transcendental meromorphic functions whose Julia sets are Jordan curves*, Proc. Amer. Math. Soc. **137** (2009), no. 11, 3733–3739. MR 2529881 (2010h:37104)
- [YPL16] Jing Yang, Shuangjie Peng, and Wei Long, *Infinitely many positive and sign-changing solutions for nonlinear fractional scalar field equations*, Discrete Contin. Dyn. Syst. **36** (2016), no. 2, 917–939. MR 3392911
- [ZXL16] Peng Zhou, Dongmei Xiao, and Yuan Lou, *Qualitative analysis for a Lotka-Volterra competition system in advective homogeneous environment*, Discrete Contin. Dyn. Syst. **36** (2016), no. 2, 953–969. MR 3392913

(PALLE E.T. JORGENSEN) DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IA 52242-1419, U.S.A.

E-mail address: palle-jorgensen@uiowa.edu

URL: <http://www.math.uiowa.edu/~jorgen/>

(FENG TIAN) DEPARTMENT OF MATHEMATICS, HAMPTON UNIVERSITY, HAMPTON, VA 23668, U.S.A.
E-mail address: `feng.tian@hamptonu.edu`